ON AN ELLIPTIC PROBLEM WITH BOUNDARY BLOW-UP AND A SINGULAR WEIGHT: THE RADIAL CASE *

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Abstract

In this work we consider the nonautonomous problem $\Delta u = a(x)u^m$ in the unit ball $B \subset \mathbb{R}^N$, with the boundary condition $u_{|\partial B} = +\infty$, and $m > 0$. Assuming that a is a continuous radial function with $a(x) \sim C_0 \ \mathrm{dist}(x, \partial B)^{-\gamma}$ as $\mathrm{dist}(x, \partial B) \to 0$, for some $C_0 > 0$, $\gamma > 0$, we completely determine the issues of existence, multiplicity and behaviour near the boundary for radial positive solutions, in terms of the values of m and γ. The case $0 < m \leq 1$, as well as estimates for solutions to the linear problem $m = 1$, are a significant part of our results.

1. Introduction

This paper is concerned with the study of some semilinear elliptic problems with boundary blow-up, of the form

$$\begin{cases} \Delta u = a(x)u^m & \text{in} \quad \Omega \\ u = +\infty & \text{on} \quad \partial\Omega \ , \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, a is a locally Hölder continuous positive function and $m > 0$. By a solution to (1.1) we understand a function $u \in C^2(\Omega)$ such that $u(x) \to +\infty$ when $d(x) := \mathrm{dist}(x, \partial\Omega) \to 0+$.

*Supported by FONDECYT projects numbers 1000627, 1000625 and 1000333 (Chile) and FEDER–MCYT project under contract BFM2001-3894 (Spain).

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The study of boundary blow up problems was originated in the work of Bieberbach [5], where $a(x) \equiv 1$, $N = 2$ and the nonlinearity u^m was replaced by e^u, in the context of Riemannian surfaces of constant negative curvature, and in the theory of automorphic functions. They were later studied under the general form $\Delta u = f(u)$ in N-dimensional domains by Keller [16] and Osserman [23]. The questions of uniqueness and asymptotic estimates for the concrete equation $\Delta u = a(x)e^u$ were first treated by [18] and [1].

Problem (1) seems to have been considered for the first time in [20] (with $a \equiv 1$ and $m = (N+2)/(N-2)$), and later in [17], [2], [3], [24], [21], [7] (with $a \in C(\overline{\Omega})$, $a > 0$ in $\overline{\Omega}$), [11], [12] (where $a = 0$ on $\partial\Omega$ is permitted) and [8], where the extension to equations involving the p-Laplacian was considered. In these works uniqueness of positive solutions is achieved through an estimate of the form $u \sim Cd(x)^{-\alpha}$ as $d(x) \to 0$. There are also some more recent papers dealing with general equations $\Delta u = f(u)$, for nonlinearities which conveniently extend $f(u) = u^m$ and $f(u) = e^u$ (see for instance [19], [4] and the list of references in [25]). On the other hand, some adaptation of these results to the field of elliptic systems have also been obtained in [6], [9], [10], [13] and [14].

In the works cited above, an important feature of the function a in (1) is that it is *bounded*. This in turn implies that $m > 1$ is a necessary and sufficient condition for the existence (and also uniqueness) of positive solutions. Our aim here is to remove this condition on a, by permitting a to be unbounded near $\partial\Omega$. For the purposes of estimates of the solutions, we are assuming that $a(x) \sim C_0 d(x)^{-\gamma}$ as $d(x) \to 0$, for some positive constants C_0 and γ. As will be seen below, this allows to have positive solutions even in the case $0 < m \le 1$.

Under this assumption on a, problem (1.1) was treated in [25], where the existence of a positive solution was obtained when $m > 1$ (also for the equation $\Delta u = a(x)e^u$), and in [22] in the context of radial solutions in a ball B (in this last one, some existence and nonexistence results concerning fairly general problems where shown). However, both in [25] and [22], no uniqueness or rate of blowup near the boundary have been obtained, and the case $0 < m \le 1$ was not considered (the nonlinearity u^m does not fulfill condition (f-1) in [22] when $0 < m \le 1$).

It is worth saying that the methods used in the literature to obtain estimates for positive solutions to problem (1.1) (see for instance [8], [7], [12] or [14]) do not work when dealing with weight functions $a(x)$ which blow up on the boundary. Thus obtaining estimates for solutions in general domains seems to be a hard task.

Hence, to gain some insight into problem (1.1), we restrict ourselves to the case where a is radial, and will mainly look for radial solutions (the case of a general domain Ω will be treated in a future work). Thus, we are dealing with

$$\begin{cases} (r^{N-1}u')' = r^{N-1}a(r)u^m & r \in (0,1) \\ u'(0) = 0, \ u(1) = +\infty \ , \end{cases} \tag{P}$$

where $m > 0$, and the weight function a will be assumed to verify throughout the following assumption:

$$\begin{cases} a \in C[0,1), \ a > 0 \text{ in } [0,1) \\ \lim\limits_{r \to 1-} (1-r)^\gamma a(r) = C_0 > 0, \ \text{for some } \gamma > 0. \end{cases} \tag{H}$$

As a first important remark, notice that this hypothesis implies in particular the existence of positive constants c, C such that

$$c(1-r)^{-\gamma} \leq a(r) \leq C(1-r)^{-\gamma},\ r \in [0,1) \tag{1.2}$$

(hypothesis (H), as well as (1.2) and estimates for all solutions below, could be expressed in terms of the even function $1-r^2$ instead of $1-r$, with a slight change of the constants involved; however, we have preferred to use $1-r$, which express the distance to the boundary ∂B).

We completely characterize the existence, uniqueness or multiplicity of solutions and estimates near the boundary for all possible radial positive solutions to (P). As an outstanding difference with the previous works, we find that in the case $m \leq 1$, there are infinitely many positive solutions, all of them with the same profile near the boundary (compare with [13]). Also, regarding the asymptotic estimates, the cases $m < 1$ and $m > 1$ seem to be completely "stable", in the sense that without further assumptions on a aside (H) we can always obtain estimates for all solutions. The linear case $m = 1$, on the contrary seems to behave very badly in this respect, since the estimates can only be obtained in general for the *logarithm* of the solutions (cf. Theorem 1.5 and Remark 1.6 below), and estimates for the solution strongly depend on the nature of second order terms in the asymptotic expansion of a near $r = 1$ (Corollary 1.7). Finally, we are able to say something about nonradial solutions in some cases.

We now state our results. The cases $m > 1$, $m < 1$ and $m = 1$ are presented separately, because they all possess some proper features of their own.

Theorem 1.1 *Assume $m > 1$ and a verifies hypothesis (H). Then problem (P) has a positive solution u if and only if $\gamma < 2$. The solution is unique and verifies*

$$\lim_{r\to 1-} (1-r)^{\alpha} u(r) = \left(\frac{\alpha(\alpha+1)}{C_0}\right)^{\frac{1}{m-1}}, \tag{1.3}$$

where $\alpha = (2-\gamma)/(m-1)$. Moreover, nonradial positive solutions to (P) do not exist.

Remark 1.2 It follows from the discussion in §3 and a continuity argument that the problem with finite positive datum n, i.e.

$$\begin{cases} \Delta u = a(x)u^m & \text{in} \quad B \\ u = n & \text{on} \quad \partial B\ , \end{cases} \tag{1.4}$$

has a unique positive solution u_n, which verifies $u_n \leq u$, where u is the solution to (P) given by Theorem 1.1. It is easy to see that $u_n \to u$ as $n \to +\infty$, uniformly in B_{r_0} for $0 < r_0 < 1$.

Theorem 1.3 *Assume $m < 1$ and a verifies hypothesis (H). If $\gamma < 2$ then problem (P) has no positive solutions, while it has infinitely many positive solutions if $\gamma \geq 2$. If $\gamma > 2$, all solutions u verify*

$$\lim_{r\to 1-} (1-r)^{\alpha} u(r) = \left(\frac{\alpha(\alpha+1)}{C_0}\right)^{\frac{1}{m-1}}, \tag{1.5}$$

where $\alpha = (2-\gamma)/(m-1)$*. When* $\gamma = 2$

$$\lim_{r\to 1-} \frac{u(r)}{(-\log(1-r))^{\beta}} = (C_0(1-m))^{\frac{1}{1-m}} \ , \tag{1.6}$$

where $\beta = 1/(1-m)$*.*

Remarks 1.4 a) In contrast with the previous case $m > 1$, it will be shown in §4.3 that *nonsymmetric* solutions to problem (P) exist when $N = 1$ (compare with [13]). This suggests that infinitely many nonradial positive solutions could exist when $N \geq 2$.

b) On the other hand, as the results in §4.2 show, problem (1.4) never has positive solutions. This two observations are applicable to the case $m = 1$ below.

Theorem 1.5 *Assume* $m = 1$ *and* a *verifies hypothesis (H). If* $\gamma < 2$ *then problem (P) has no positive solutions, while it has infinitely many positive solutions if* $\gamma \geq 2$*. All solutions* u *are constant multiples of each other and verify*

$$\lim_{r\to 1-} (1-r)^{\delta} \log u(r) = \frac{\sqrt{C_0}}{\delta} \tag{1.7}$$

when $\gamma > 2$*, with* $\delta = (\gamma-2)/2$ *and*

$$\lim_{r\to 1-} \frac{\log u(r)}{-\log(1-r)} = \sigma \tag{1.8}$$

when $\gamma = 2$*, where* $\sigma = (-1+\sqrt{1+4C_0})/2$*.*

Remarks 1.6 A natural question to ask in the linear case is the following: can we obtain estimates for the behaviour of u (as in Theorems 1 and 2) instead of $\log u$? The following examples show that the growth of positive solutions near $r = 1$ can be arbitrarily prescribed, if the weight a is to satisfy only hypothesis (H).

a) The function $u(r) = (1-r)^{-\sigma}\xi(1-r)$, where $\sigma = (-1+\sqrt{1+4C_0})/2$, solves the one-dimensional version of (P) with the weight

$$a(r) = \frac{C_0}{(1-r)^2} - \frac{2\sigma}{1-r}\frac{\xi'(1-r)}{\xi(1-r)} + \frac{\xi''(1-r)}{\xi(1-r)} \ .$$

Thus if $\xi(t)$ is positive for t near zero and verifies $t^2\xi''(t) - 2\sigma t\xi'(t) = o(\xi(t))$ as $t \to 0+$, a fulfills condition (H) (note that we are only interested in positivity near $r = 1$). Particular examples are $\xi(t) = (\log t)^{\tau}$, $\tau > 0$ and $\xi(t) = \log(\log t)$.

b) In the same manner, the function $u(r) = e^{\sqrt{C_0}(1-r)^{-\delta}/\delta}\xi(1-r)$ solves (P) with $N = 1$ and

$$a(r) = \frac{C_0}{(1-r)^{\gamma}} + \frac{(\delta+1)\sqrt{C_0}}{(1-r)^{\delta+2}} - \frac{2\sqrt{C_0}}{(1-r)^{\delta+1}}\frac{\xi'(1-r)}{\xi(1-r)} + \frac{\xi''(1-r)}{\xi(1-r)} \ .$$

Thus if $(\delta+1)\sqrt{C_0}t^{\gamma/2-1}\xi(t) - 2\sqrt{C_0}\delta^{-1}t^{\gamma/2}\xi'(t) + t^{\gamma}\xi''(t) = o(\xi(t))$ as $t \to 0+$, a verifies condition (H). Again the functions $\xi(t) = (\log t)^{\tau}$, $\tau > 0$ and $\xi(t) = \log(\log t)$ are valid.

To correct this "anomaly", we need to impose some condition on the "second order terms" of the weight a near $r = 1$. In the light of the former examples, it seems to us that condition (1.9) below is optimal. Notice that considering (1.7), the estimate for $\gamma > 2$ is not exactly what one would expect.

Corollary 1.7 *Assume a verifies hypothesis (H), together with*

$$\int_0^1 (1-r)\left|a(r) - \frac{C_0}{(1-r)^\gamma}\right| \, dr < +\infty \ . \tag{1.9}$$

Then every positive solution u to (P) with $m = 1$ verifies

$$\lim_{r\to 1-} (1-r)^\sigma u(r) = \theta \tag{1.10}$$

if $\gamma = 2$, where $\sigma = (-1 + \sqrt{1+4C_0})/2$ and

$$\lim_{r\to 1-} \frac{u(r)}{(1-r)^{\frac{\delta+1}{2}} e^{\sqrt{C_0}(1-r)^{-\delta}/\delta}} = \theta \tag{1.11}$$

for $\gamma > 2$, where $\delta = (\gamma - 2)/2$ and θ is some positive constant.

Remark 1.8 A "natural" way to proceed in view of Theorem 1.9 below is to solve problem (P) with $a(r) = C_0(1-r)^{-\gamma}$. However, in the simplest case $N = 1$, it turns out that the point $r = 1$ is of singular irregular type, and the solutions could be in principle very badly behaved in a neighbourhood of $r = 1$. Actually, except for a few exceptional γ's for which the solution is an explicit finite combination of powers and exponentials, this equation can be solved in terms of power series of singular terms (see p. 401 in [15]) and this makes it almost impossible to determine the behaviour of the blow-up solutions near $r = 1$. This behaviour is obtained indirectly by means of Corollary 1.7. We quote that in the case $\gamma = 4$, the general solution can be explicitly obtained by means of the rescaling $v(r) = (1-r)u((1-r)^{-1})$.

Corollary 1.7 is a consequence of the following (general) sturmian reminiscent comparison theorem, which is interesting in its own right.

Theorem 1.9 *Let a, b be continuous functions in $[r_0, 1)$ for some $0 < r_0 < 1$, and assume*

$$\int_{r_0}^1 (1-r)|a(r) - b(r)| dr < +\infty \ .$$

Let u, v be positive solutions to $(r^{N-1}u')' = r^{N-1}a(r)u$ and $(r^{N-1}v')' = r^{N-1}b(r)v$ in $(r_0, 1)$, respectively, which are nondecreasing for r close to 1. Then

$$\lim_{r\to 1-} \frac{u(r)}{v(r)} = \theta \ ,$$

where θ is some positive constant.

We finally remark that our proofs of existence and nonexistence (and multiplicity of positive solutions when $0 < m \leq 1$) apply to some more general weight functions $a(r)$. Also, the nonlinearity u^m could be replaced by a continuous increasing function $f(u)$, and some more general operators (like the p-Laplacian) could be considered.

The paper is organized as follows: Section 2 contains some preliminary lemmas. To make the exposition clear, we have distributed the cases $m > 1$, $m < 1$ and $m = 1$ in Sections 3, 4 and 5 respectively. Also, in each section the issues of existence, uniqueness or multiplicity and estimates are treated separately.

2. Preliminary Lemmas

To carry out our analysis of the positive solutions to (P), it is natural to consider the solutions to the following Cauchy problem,

$$\begin{cases} (r^{N-1}u')' = r^{N-1}a(r)u^m & r \in (0,1) \\ u(0) = u_0, \; u'(0) = 0 \; , \end{cases} \tag{C}$$

for $u_0 > 0$. The next Lemma will be the basis of the later developments. We omit the proof, which is a consequence of rather standard arguments.

Lemma 2.1 *Assume $m > 0$ and a verifies (H). Then for every $u_0 > 0$, there exists a unique solution u to (C), defined in an interval $[0,\omega)$ with $\omega \leq 1$. This solution is positive and increasing, and if $u_0 < v_0$, the corresponding solutions $u(r)$, $v(r)$ verify $u(r) < v(r)$ in the interval of definition of v.*

We also quote that, by direct integration, solutions to (C) verify the following equation for $0 \leq r_0 \leq r < \omega$

$$u(r) = u(r_0) + r_0^{N-1}u'(r_0)\left(\frac{r_0^{-(N-2)} - r^{-(N-2)}}{N-2}\right) + \int_{r_0}^{r}\int_{r_0}^{t}\left(\frac{s}{t}\right)^{N-1} a(s)u(s)^m \, ds \, dt \; , \tag{2.1}$$

in the case $N \geq 3$, and

$$u(r) = u(r_0) + r_0u'(r_0)\log\left(\frac{r}{r_0}\right) + \int_{r_0}^{r}\int_{r_0}^{t}\left(\frac{s}{t}\right)^{N-1} a(s)u(s)^m \, ds \, dt \; , \tag{2.2}$$

for $N = 2$.

An important consequence of (2.1) and (2.2) is the next Lemma, that can also be proved in a standard way. It states that a supersolution can never be reached by a solution from above, nor a subsolution from below.

Lemma 2.2 *Assume $m > 0$ and a verifies (H). Then if u is a solution to (C) and $\bar{u}$ a supersolution verifying $\bar{u}(0) < u(0)$, we have $\bar{u}(r) < u(r)$, whenever this inequality makes sense. Similarly, if $\underline{u}$ is a subsolution with $\underline{u}(0) > u(0)$, then $\underline{u}(r) > u(r)$.*

After these preliminaries we can begin to analyze all the different cases that can occur for problem (P). We divide the study in several sections and subsections for the sake of clarity.

3. The case $m > 1$

In this case the issues of existence and uniqueness are not changed by the appearance of a singularity in a. The main difference in our situation is the rate of blow up in the boundary, and we have to say that comparison techniques which have been used frequently before seem to be useless in our problem.

3.1. Nonexistence for $\gamma \geq 2$. Assume that we have a solution u to (P). Choose $r_0 \in (0,1)$. Then for $r \in (r_0, 1)$, we have

$$(r^{N-1}u')' \geq r^{N-1}C(1-r_0)^{-\gamma}u^m ,$$

for some positive constant C. Putting $v(r) = (C(1-r_0)^{2-\gamma})^{\frac{1}{m-1}}u(r_0 + (1-r_0)r)$, we obtain that $(r^{N-1}v')' \geq r^{N-1}v^m$ in $(0,1)$, that is, v is a subsolution to the equation $(r^{N-1}V')' = r^{N-1}V^m$ such that $v(1) = +\infty$. Let V be the unique solution to this equation with $V'(0) = 0$, which blows up at $r = 1/2$ (see [3]). We claim that $v \leq V$. Indeed, $W = v - V$ satisfies $W'(0) > 0$ and $W(1/2) = -\infty$. Thus W has an interior maximum in $(0, 1/2)$. At this maximum, $(r^{N-1}W')' \leq 0$, which leads to $W \leq 0$. Thus $v \leq V$. We then conclude:

$$u(r_0 + (1-r_0)r) \leq (C^{-1}(1-r_0)^{\gamma-2})^{\frac{1}{m-1}}V(r) \quad \text{in } (0, 1/2) . \tag{3.1}$$

Setting, for example, $r = 0$ and then letting $r_0 \to 1$, we obtain that u is bounded at $r = 1$, which is impossible. Notice that this argument shows indeed that for $\gamma > 2$ all solutions to (C) are defined in an interval $[0, w)$, with $w < 1$. □

3.2. Existence and estimates for $\gamma < 2$. We prove existence by using the method of sub and supersolutions. This method is also valid for solutions which blow-up on the boundary, as shown in Lemma 4 of [12]. We take as a supersolution the function $\bar{u} = \Lambda(1-r^2)^{-\alpha}$. An easy calculation shows

$$\Delta\bar{u} - a(r)\bar{u}^m \leq \Lambda(1-r^2)^{-\alpha-2}(4r^2\alpha(\alpha+1) + 2\alpha N(1-r^2) - c\Lambda^{m-1}) \leq 0 ,$$

provided that Λ is taken big enough, where we have used (1.2). Similarly, $\underline{u} = \lambda(1-r^2)^{-\alpha}$ is a subsolution if λ is small enough. Thus problem (P) has at least a positive solution, verifying $\underline{u} \leq u \leq \bar{u}$.

Now let us see that the inequality $\underline{u} \leq u \leq \bar{u}$ is indeed valid for every positive solution u to (P), with a convenient choice of λ and Λ. Notice that (3.1) holds when $m > 1$, regardless of the values of γ. In particular, we have $u \leq \Lambda(1-r)^{-\alpha}$.

To obtain the lower estimate, we multiply the equation in (P) by u' and integrate in (r_0, r) for r_0 close to 1. Using (1.2), we arrive at

$$\frac{u'(r)}{\sqrt{u(r)^{m+1} - u(r_0)^{m+1}}} \leq \frac{u'(r_0)}{\sqrt{u(r)^{m+1} - u(r_0)^{m+1}}} + C(1-r)^{-\gamma/2} \leq C'(1-r)^{-\gamma/2} ,$$

where C and C' stand for positive constants. Integrating this last inequality in $(r_0, 1)$, we have

$$\int_{u(r_0)}^{+\infty} \frac{d\tau}{\sqrt{\tau^{m+1} - u(r_0)^{m+1}}} \leq C'(1-r_0)^{-\frac{\gamma-2}{2}}$$

(the integral is convergent since $m > 1$). Performing the change of variables $\tau = u(r_0)s$ in the integral leads to $u(r_0) \geq \lambda(1-r_0)^{-\alpha}$, where

$$\lambda = \left(\frac{1}{C'} \int_1^{+\infty} \frac{ds}{\sqrt{s^{m+1}-1}} \right)^{\frac{2}{m-1}} .$$

This inequality is valid in $[0,1)$ (by diminishing λ if necessary). This shows $\underline{u} \leq u \leq \bar{u}$.

Our next step is proving that every positive solution u to (P) verifies (1.3). For this sake, we introduce the change of variables:

$$\rho := \begin{cases} \dfrac{1}{N-2}\left(1 - \dfrac{1}{r^{N-2}}\right), & N \geq 3 \\ \log r, & N = 2\ , \end{cases} \tag{3.2}$$

which transforms our equation into $u'' = g(\rho)a(\rho)u^m$ in the interval $(-\infty, 0)$, where $g(\rho) = r^{2(N-1)}$, and derivatives are taken with respect to ρ (with a slight abuse of notation we use $a(\rho)$ instead of $a(r(\rho))$). The asymptotic behaviour of a gives that $\lim_{\rho\to 0-} g(\rho)a(\rho)(-\rho)^\gamma = C_0$. If we set $u = (-\rho)^{-\alpha}v$, the resulting equation for v can be written in the form

$$((-\rho)^{-2\alpha}v')' = (-\rho)^{-2(\alpha+1)}(g(\rho)a(\rho)(-\rho)^\gamma v^m - \alpha(\alpha+1)v).$$

Recall that $0 < \lambda \leq v \leq \Lambda$. The further change of variables $t = -\log((-\rho)^{2\alpha+1}/(2\alpha+1))$ leads to

$$v'' + v' = K(c(t)v^m - v) \qquad \text{in } (-\infty, \infty) \tag{3.3}$$

where $K = \alpha(\alpha+1)/(2\alpha+1)^2$, $c(t) = (\alpha(\alpha+1))^{-1}g(\rho)a(\rho)(-\rho)^\gamma$. Since $\lim_{t\to+\infty} c(t) = C_0/(\alpha(\alpha+1)) =: c_\infty$, in order to prove (1.3) it is enough to show that $\lim_{t\to+\infty} v(t) = c_\infty$.

In order to prove this, we assume first that v is monotone for large t. Then v has a finite positive limit. Consequently the right-hand side in (3.3) also has a limit which we denote by θ. We claim that $\theta = 0$. Indeed if we assume $\theta > 0$, then $v'' + v' \geq \theta/2$ for large t. A direct integration then implies $v \to +\infty$ as $t \to +\infty$, contradicting the fact that v is bounded. The case $\theta < 0$ can be ruled out in a similar way. Hence $\theta = 0$, and since v is bounded away from zero, it follows that $\lim_{t\to+\infty} v(t) = c_\infty$.

It only remains to consider the case when v' changes sign infinitely many times. Then there exists a sequence of points $\{t_n\}$ such that $v(t_{2n})$ is a local maximum and $v(t_{2n+1})$ a local minimum. Since $v'(t_{2n}) = v'(t_{2n+1}) = 0$, $v''(t_{2n}) \leq 0$, $v''(t_{2n+1}) \geq 0$, (3.3) implies

$$c(t_{2n+1})^{-\frac{1}{m-1}} \leq v(t_{2n+1}) \leq v(t_{2n}) \leq c(t_{2n})^{-\frac{1}{m-1}}.$$

We conclude that $\lim_{n\to+\infty} v(t_n) = c_\infty$. It follows that $\lim_{t\to+\infty} v(t) = c_\infty$. This finishes the proof of (1.3). $\square$

3.3. Uniqueness. Let u, v be arbitrary positive solutions to (P). If we denote $w = u/v$, then $w = 1$ in $r = 1$, as the results in §3.2 show. Assume that the set $B^+ := \{w > 1\}$ is nonempty. Then

$$v\Delta w + 2\nabla v \nabla w = a(r)v^m w(w^{m-1} - 1) > 0$$

in B^+, and the maximum principle gives $w(r) \le 1$ in B^+, a contradiction. This proves that B^+ is empty and so $w \le 1$. A similar argument gives $w \ge 1$, and uniqueness gets proved. □

3.4. Nonexistence of nonradial solutions. Assume v is an arbitrary solution to (1.1). Then by comparison it follows that $v \ge u_n$, where u_n is the unique positive solution to (1.4) (see Remark 1.2). This leads to $v \ge u$, the unique solution to (P) given by Theorem 1.1.

On the other hand, let $0 < r_n < 1$ such that $r_n \to 1$ as $n \to +\infty$, and denote by v_n the (radial) solution to

$$\begin{cases} \Delta u = a(x) u^m & \text{in} \quad B_n \\ u = +\infty & \text{on} \quad \partial B_n \ , \end{cases}$$

where B_n stands for the ball with radius $0 < r_n < 1$. This solution is already known to be unique since the weight a is continuous in $\overline{B_n}$ (cf. [3]). Since v is a subsolution to this problem, we have $v \le v_n$. However, as $n \to +\infty$, we obtain that $v_n \to u$ (by uniqueness) and so $v \le u$, which gives $u = v$. □

4. The case $m < 1$

This case (which somehow will be found to be similar to $m = 1$) presents some very interesting features of its own. Namely, we can prove that problem (P) has infinitely many positive solutions, all of them with the same asymptotic profile near the boundary.

4.1. Nonexistence of solutions with $0 < \gamma < 2$. Notice that, since $a(r)$ is regular away from $r = 1$ and $m < 1$, it is standard to conclude that the solutions u to (C) are defined at least in $[0, 1)$. Let us see that $\lim_{r \to 1-} u(r)$ is always finite. Assume first $\gamma \ne 1$. Using (2.1) and (2.2) with $r_0 = 0$, and denoting $u_\delta = \sup_{[0,\delta]} u(r)$, we obtain

$$u_\delta \le u_0 + \frac{C u_\delta^m}{\gamma - 1} \left(\frac{1 - (1-r)^{2-\gamma}}{2 - \gamma} - r \right) \ ,$$

where we have used (1.2). Since $\gamma < 2$ and $m < 1$, we conclude that u_δ is always bounded. When $\gamma = 1$, the only difference is that

$$u_\delta \le u_0 + C u_\delta^m ((1-r)\log(1-r) + r) \ ,$$

and the conclusion is the same. □

4.2. Existence and estimates for solutions. Assume for the moment $\gamma > 2$. Just like in §3.2, it can be proved that $\underline{u} = \lambda(1 - r^2)^{-\alpha}$ and $\bar{u} = \Lambda(1 - r^2)^{-\alpha}$ are sub and supersolution respectively, but now for large λ and small Λ (thus $\bar{u} \le \underline{u}$). Lemma 2.2 implies that if u is a solution to (C) with $\bar{u}(0) < u(0) < \underline{u}(0)$, then $\bar{u}(r) < u(r) < \underline{u}(r)$, $r \in [0, 1)$. This in particular shows that u is a solution to (P). Since $u(0)$ is arbitrary and λ and Λ can be chosen arbitrarily large and small, respectively, we conclude that all solutions to (C) are solutions to (P).

We now turn to the estimate (1.5). Notice that we already know that $A := \liminf_{r \to 1-} (1 - r)^\alpha u(r)$ and $B := \limsup_{r \to 1-} (1 - r)^\alpha u(r)$ are positive and finite. The inequalities

$$A \ge \left(\frac{C_0}{\alpha(\alpha+1)} \right)^{\frac{1}{1-m}} , \ B \le \left(\frac{C_0}{\alpha(\alpha+1)} \right)^{\frac{1}{1-m}} \ ,$$

can be obtained through the procedure used in the case $\gamma = 2$ below, so we are not showing the details. Estimate (1.5) is a direct consequence of these two inequalities.

Now let $\gamma = 2$. It is easily checked that $\bar{u} = \Lambda(-\log(1-r))^\beta$ is a supersolution when Λ is small enough, and $\underline{u} = \lambda(-\log(1-r))^\beta$ is a subsolution for λ large. Thus the same reasoning as before proves that every solution to (C) is indeed a solution to (P). To prove estimate (1.6) we let

$$A := \liminf_{r\to 1-} \frac{u(r)}{(-\log(1-r))^\beta} > 0 ,$$

and we are going to show that $A \geq (C_0(1-m))^{\frac{1}{1-m}}$. Let $\varepsilon > 0$ be small and r_0 close to 1 so that $u(r) \geq (A-\varepsilon)(-\log(1-r))^\beta$ and $a(r) \geq (C_0-\varepsilon)(1-r)^{-2}$ if $r_0 \leq r < 1$. Then

$$u(r) \geq O(1) + (C_0+\varepsilon)(A+\varepsilon)^m \int_{r_0}^r \int_{r_0}^t (1-s)^{-2}(-\log(1-s))^{\beta m} ds\ dt . \tag{4.1}$$

Denote this integral by $I(r)$. A change of variables together with integration by parts gives

$$I(r) = O(1) + (1-m)(-\log(1-r))^\beta - \beta m \int_{r_0}^r \int_{r_0}^t (1-s)^{-2}(-\log(1-s))^{\beta m-1} dsdt$$

$$\geq O(1) + (1-m)(-\log(1-r))^\beta + \frac{\beta m}{\log(1-r_0)} I(r) .$$

Thus coming back to (4.1), we obtain after letting $r \to 1$,

$$A \geq \frac{(C_0-\varepsilon)(A-\varepsilon)^m(1-m)}{1-\frac{\beta m}{\log(1-r_0)}} .$$

If we make $r_0 \to 1$ and $\varepsilon \to 0$, we arrive at $A \geq (C_0(1-m))^{\frac{1}{1-m}}$. The reversed inequality for the upper limit is obtained similarly. This completes the proof. □

4.3. Construction of nonsymmetric solutions (Remarks 1.4 a)). Let us see that for the one-dimensional version of (P) infinitely many nonsymmetric solutions can be constructed. Indeed, consider the initial value problem

$$\begin{cases} u'' = a(r)u^m & r \in (0,1) \\ u(0) = u_0,\ u'(0) = \eta , \end{cases} \tag{4.2}$$

for $u_0 > 0$, $\eta \neq 0$. It follows in a standard fashion that problem (4.2) has a unique solution u_η which is continuous with respect to η (we will consider u_0 as fixed). Thus, for any interval $[-r_0, r_0]$, we can select η small so that u_η is conveniently close to u, solution to (4.2) with the same value of u_0 and $\eta = 0$. In particular, u_η will be increasing and convex outside a neighbourhood of $r = 0$. Now since the equation is sublinear, we can continue the solution u_η to the whole interval $(-1,1)$, and also $\lim_{r\to\pm 1} u_\eta(r) = +\infty$ (compare with §5.1). To summarize, u_η is always a solution to (P), which is not symmetric. □

5. The linear problem

This last case is a complete analogue to $m < 1$ concerning the issues of existence and multiplicity of positive solutions. The only point worth stressing is the difference in nature of the asymptotic estimates near $r = 1$.

5.1. Existence and nonexistence of solutions. Let us see first that (P) has no positive solutions if $0 < \gamma < 2$. The proof is similar to that in §4.1. By linearity of the equation in (C) and continuity of $a(r)$ in $[0, 1)$, we deduce that all solutions to (C) are defined in the interval $[0, 1)$. Now we are showing that they remain bounded as $r \to 1-$. Assume $N \geq 3$, the case $N = 2$ being treated in the same way. Setting $u_\delta = \sup_{[r_0, r_0+\delta]} u(r)$ and using (2.1), we arrive at

$$u_\delta \leq u(r_0) + \frac{1}{N-2}\, r_0 u'(r_0) + u_\delta \int_{r_0}^{r} \int_{r_0}^{t} a(s)\ ds\ dt\ .$$

Since this last integral converges if $\gamma < 2$, we can select r_0 close to 1 so that it becomes less than $1/2$. Thus we have a bound for u_δ, and (P) can never have positive solutions.

Now let $\gamma \geq 2$. As quoted before, all solutions are defined in $[0, 1)$. Let us see that $\lim_{r\to 1-} u(r) = +\infty$. Indeed, this is an easy consequence of

$$u(r) \geq u(0) + \frac{Cu(0)^m}{\gamma - 1}\left(\frac{(1-r)^{-(\gamma-2)}}{\gamma - 2} - r\right)$$

when $\gamma > 2$ or

$$u(r) \geq u(0) - Cu(0)^m(\log(1-r) + r)$$

for $\gamma = 2$, which are obtained from (2.1) and (2.2) with $r_0 = 0$. Finally, notice that the linearity of the equation and the uniqueness of solutions to (C) (see Lemma 2.1) imply that all solutions are constant multiples of each other. □

5.2. Estimates (1.7) and (1.8). Let u be a solution to (P), and perform the change of variables given by (3.2). In this way we consider now the one-dimensional problem $u'' = g(\rho)a(\rho)u$, $\rho \in (-\infty, 0)$. Let $v = \log u$, and $v' = z$. Then z is a solution to

$$z' + z^2 = g(\rho)a(\rho), \quad \rho \in (-\infty, 0)\ . \tag{5.1}$$

Let $z = (-\rho)^{-\gamma/2} w$ (notice that $w > 0$, since u is increasing). We have for w the following equation:

$$(-\rho)^{\gamma/2} w' = g(\rho)a(\rho)(-\rho)^\gamma - w^2 - \frac{\gamma}{2}(-\rho)^{\frac{\gamma-2}{2}} w\ .$$

If we set

$$w_0(\rho) = -\frac{\gamma}{4}(-\rho)^{\frac{\gamma-2}{2}} + \frac{1}{2}\sqrt{\frac{\gamma^2}{4}(-\rho)^{\gamma-2} + 4g(\rho)a(\rho)(-\rho)^\gamma}$$

then it is clear that when $w(\rho) > w_0(\rho)$, w is decreasing, while it is increasing when $w(\rho) < w_0(\rho)$. Thus it follows that

$$\lim_{\rho\to 0-} w(\rho) = \lim_{\rho\to 0-} w_0(\rho) = \begin{cases} \sqrt{C_0}, & \gamma > 2 \\[2ex] \dfrac{-1 + \sqrt{1 + 4C_0}}{2}, & \gamma = 2\ . \end{cases}$$

Finally, estimates (1.7) and (1.8) are a consequence of l'Hôpital rule. □

Remark 5.3 Equation (5.1) is a Riccati equation, which can not be solved in general. However, when we set in the right hand-side $C(-\rho)^{-2}$, $C>0$, we have the explicit solution:

$$z(\rho) = A(-\rho)^{-1} + \frac{2A+1}{\rho + (2A+1)\lambda(-\rho)^{-2A}} \ ,$$

for $\lambda \in \mathbb{R}$ arbitrary, and $A(A+1) = C$. By comparison we could use this solution to prove estimate (1.8) ($\gamma = 2$).

5.3. Proof of Corollary 1.7. The function $v(r) = (1-r)^{-\sigma}$ is a solution to $(r^{N-1}v')' = r^{N-1}b(r)v$ in $(1/2, 1)$, where

$$b(r) = \frac{C_0}{(1-r)^2} + \frac{N-1}{r}\sigma\frac{1}{(1-r)} \ .$$

Thus (1.10) follows from Theorem 1.9 and (1.9). This proves the case $\gamma = 2$.

The case $\gamma > 2$ is not so straightforward. For simplicity we consider $N = 1$ only, since the N dimensional case can be treated with minor variations of the argument. The idea is to select a weight $b(r)$ so that $(1-r)|b(r) - C_0(1-r)^{-\gamma}|$ is integrable and the equation $(r^{N-1}v')' = r^{N-1}b(r)v$ can be solved. For this sake we proceed inversely, and let $v(r) = e^{C(1-r)^{-\delta}+z(r)}$, where z is to be determined and $C = \sqrt{C_0}/\delta$. It is easy to see that

$$(1-r)(b(r) - C_0(1-r)^{-\gamma}) = C\delta(\delta+1)(1-r)^{-\delta-1} + 2C\delta(1-r)^{-\delta}z' + (1-r)(z')^2 + (1-r)z''.$$

In order to make this function absolutely integrable, we could think of choosing z' so that the first two terms vanish, that is $z' = -(\delta+1)(1-r)^{-1}/2$, but this will lead to

$$(1-r)(b(r) - C_0(1-r)^{-\gamma}) = \frac{\delta^2-1}{4}(1-r)^{-1} \ ,$$

which is only integrable if $\delta = 1$, i. e. $\gamma = 4$ (see Remark 1.8). However, the behaviour of this term is better than before, and this leads us to consider instead $z' = -(\delta+1)(1-r)^{-1}/2 + h'$, getting that

$$(1-r)(b(r) - C_0(1-r)^{-\gamma}) = \frac{\delta^2-1}{4}(1-r)^{-1} + 2C\delta(1-r)^{-\delta}h' + (1-r)(h')^2 - (\delta+1)h' + (1-r)h'' \ .$$

With the final choice $h' = -(\delta^2-1)(1-r)^{\delta-1}/8C\delta$, we obtain that

$$(1-r)(b(r) - C_0(1-r)^{-\gamma}) = \frac{(\delta^2-1)^2}{64C^2\delta^2}(1-r)^{2\delta-1} + \frac{\delta^2-1}{4C}(1-r)^{\delta-1}$$

is always absolutely integrable, since $\delta > 0$. To summarize, the function $v(r) = (1-r)^{\frac{\delta+1}{2}}e^{C(1-r)^{-\delta}}e^{(\delta^2-1)(1-r)^{\delta}/8C}$ solves $v'' = b(r)v$, and condition (1.9) together with Theorem 1.9 give that for every positive solution u to (P) with $N = 1$

$$\lim_{r\to 1-} \frac{u(r)}{v(r)} = \theta \ ,$$

for some positive constant θ, but this is precisely (1.11). This proves the Corollary. $\square$

5.4. Proof of Theorem 1.9. Introducing the change of variables given by (3.2), we can consider the problems $u'' = g(\rho)a(\rho)u$, $v'' = g(\rho)b(\rho)v$ in some interval $(l, 0)$, $l < 0$, where g is positive, increasing and continuous with $g(0) = 1$. We fix $l < \rho_0 < 0$ and set

$$F(\rho) = \int_{\rho_0}^{\rho} \frac{1}{v(s)^2} ds, \quad G(\rho) = F^{-1}(\rho) \ .$$

Then F is a differentiable, increasing positive function with values in $[0, d]$, for some $d > 0$. Similarly, G is differentiable, increasing and negative, with values in $[\rho_0, 0]$. For $r \in (0, d)$ we define the function

$$w(r) = \frac{u(G(r))}{v(G(r))} \ .$$

A little algebra shows that w is a solution to the equation

$$w'' = g(G(r))(a(G(r)) - b(G(r)))v(G(r))^4 w(r), \ r \in (0, d) \ .$$

Integrating twice this equation in the interval $(0, r)$ and letting $w_\delta = \sup_{[0,\delta]} w(r)$, we obtain

$$w(r) \le w(0) + |w'(0)|\delta + w_\delta \int_0^r \int_0^t g(G(\tau))|a(G(\tau)) - b(G(\tau))|v(G(\tau))^4 \ d\tau \ dt.$$

An application of Fubini's theorem together with the change of variables $\sigma = G(\tau)$ leads to

$$w(r) \le w(0) + |w'(0)|\delta + w_\delta \int_{\rho_0}^{G(r)} (d - F(\sigma))g(\sigma)|a(\sigma) - b(\sigma)|v(\sigma)^2 \ d\sigma \ .$$

Notice that $d = F(0)$ and $v' \ge 0$, so it can be shown by l'Hôpital rule that $(d - F(\sigma))v(\sigma)^2 \le C(-\sigma)$. Thus the integral in the right-hand side of the above formula converges as $r \to d$ by (1.9) (we recall that $g(0) = 1$), and we can take ρ close enough to zero so that it is less than 1/2. This entails an upper bound for w. Proceeding similarly with $\tilde{w}(r) = v(G(r))/u(G(r))$ we get a lower bound.

Thus it only remains to show that w has a limit when $r \to d-$. Assume that there exist infinite sequences of points $\{s_n\}$ and $\{t_n\}$ converging to d, such that $s_n < t_n$, and w attains local maxima and minima in s_n and t_n, respectively. Then, integrating the equation for w in (s_n, t_n), we have

$$w(s_n) - w(t_n) \le M \int_{s_n}^{t_n} \int_0^t g(G(\tau))|a(G(\tau)) - b(G(\tau))|v(G(\tau))^4 \ d\tau \ dt,$$

where $M = \sup w$. The last integral tends to zero as $n \to \infty$, by (1.9) and so the claim follows. This proves the theorem. $\square$

Acknowledgements

The authors would like to thank the referee for several helpful comments, which improved the manuscript.

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